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An Entire Holomorphic Function Associated to an Entire Harmonic Function

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Let h be a harmonic function on $\mathbb{R}^N$. Then there exists a holomorphic function f on $\mathbb{C}$ such that $f(t) = h(t, 0, ..., 0)$ for all real t. Precise inequalities relating the growth rate of f to that of h are proved. These results are applied to deduce uniqueness theorems for harmonic functions of sufficiently slow growth that vanish at certain lattice points. Another application concerns the rate at which a harmonic function of finite order can decay along a ray.

1. INTRODUCTION

If h is a function that is harmonic on the whole of the Euclidean space $\mathbb{R}^N$, where $N \geqslant 2$, then there is a unique entire (holomorphic) function f on the complex plane $\mathbb{C}$ such that $f(t) = h(t, 0, ..., 0)$ for all real t. This fact has been used to deduce theorems for harmonic functions on $\mathbb{R}^N$ from classical results about entire functions (see, e.g., [2, 17, 20]). Inequalities between the growth rates of h and f are crucial for the success of the technique. In this paper we prove such inequalities which are more comprehensive and precise than those obtained hitherto. We then apply these results to deduce two groups of uniqueness theorems for harmonic functions. Theorems in the first group assert that harmonic functions of sufficiently slow growth are uniquely determined by their values at certain lattice points in $\mathbb{R}^N$ (see Sections 3, 4 below); these results are compared with those obtained in [2, 8, 13, 20, 25]. A typical result in the second group says that a harmonic function of finite order cannot decay rapidly on a ray unless it is identically zero on the ray (see Sections 5, 6). This contrasts with the fact that a harmonic function of unrestricted growth can decay arbitrarily rapidly on almost all rays emanating from the origin ([5]; for $N = 2$ see also [21]).

Now we introduce some notation. The space of functions that are harmonic on $\mathbb{R}^N$ is denoted by $\mathscr{H}_N$ and the space of entire functions on $\mathbb{C}$

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is denoted by $\mathscr{E}$. If $f \in \mathscr{H}_N$ (respectively, $\mathscr{E}$), then we write $M_\infty(f, r)$ for the maximum value of $|f|$ on the sphere (respectively, circle) of radius r centred at the origin. As usual, the *order* $\rho(f)$ of f is defined by

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log \log M_\infty(f, r)}{\log r}$$

if f is non-constant; by convention, the order of a constant function is 0. Thus $0 \leqslant \rho(f) \leqslant +\infty$. If $0 < \rho(f) < +\infty$, then the *type* $\tau(f)$ of f is defined by

$$\tau(f) = \limsup_{r \to +\infty} r^{-\rho(f)} \log M_\infty(f, r),$$

so that $0 \leqslant \tau(f) \leqslant +\infty$; if $\rho(f) = 0$ or $+\infty$, then $\tau(f)$ is undefined. We say that f is of *growth* (ρ, τ) if $\rho(f) < \rho$ or $\rho(f) = \rho$, $\tau(f) \leqslant \tau$. Sometimes it is more convenient to measure growth in terms of the L^2 mean defined by

$$M_2(f, r) = \left(\int_S |f(rx)|^2 \, d\sigma(x) \right)^{1/2},$$

where S is the unit sphere in $\mathbb{R}^N$ (or the unit circle in $\mathbb{C}$) and σ is $(N-1)$-dimensional surface measure (or length measure) normalized so that $\sigma(S) = 1$. The values of $\rho(f)$ and $\tau(f)$ are unaffected if $M_\infty(f, r)$ is replaced by $M_2(f, r)$ in their definitions (see [14, Lemma 2.2] for the harmonic case).

We shall often use C to denote a positive constant, not necessarily the same on any two occurrences. To indicate that C depends on $a, b, \ldots$, we write $C = C(a, b, \ldots)$.

Proposition 1. *If $h \in \mathscr{H}_N$, then there exists a unique function $f \in \mathscr{E}$ such that*

$$f(t) = h(t, 0, \ldots, 0) \tag{1.1}$$

for all real t, and if h is of growth (ρ, τ), then so also is f.

In the case where $\rho(h) = 1$, Proposition 1 is essentially contained in [20, Corollary 1.5]. We next give a much more precise result for certain functions of order 1.

Proposition 2. *If $h \in \mathscr{H}_N$ and*

$$M_2(h, r) = O(r^p e^{\lambda r}) \qquad (r \to +\infty) \tag{1.2}$$

for some real number p and positive number λ, then the entire function f satisfying (1.1) *is such that*

$$M_\infty(f, r) = O(r^{p+(2N-3)/4} e^{\lambda r}) \qquad (r \to +\infty). \tag{1.3}$$

The result remains true if O is replaced by o in both (1.2) *and* (1.3).

The result is best possible in the sense that it becomes false if O is replaced by o in (1.3) *but not in* (1.2).

2. PROOFS OF PROPOSITIONS 1 AND 2

2.1. Let $\mathscr{H}_{m,N}$ denote the vector space of all homogeneous harmonic polynomials of degree m on $\mathbb{R}^N$. Suppose that $h \in \mathscr{H}_N$. Then h has a unique expansion of the form $h = \sum_{j=0}^{\infty} H_j$, where $H_j \in \mathscr{H}_{j,N}$, and the series $\sum_{j=0}^{\infty} |H_j|$ is locally uniformly convergent on $\mathbb{R}^N$ (see, e.g., [6, p. 84]). We call $\sum_{j=0}^{\infty} H_j$ the *polynomial expansion* of h. Writing $\mathbf{e}$ for the vector $(1, 0, ..., 0)$ in $\mathbb{R}^N$, we have

$$h(t\mathbf{e}) = \sum_{j=0}^{\infty} H_j(t\mathbf{e}) = \sum_{j=0}^{\infty} H_j(\mathbf{e})\, t^j$$

for all real t. Let

$$f(z) = \sum_{j=0}^{\infty} H_j(\mathbf{e})\, z^j. \tag{2.1}$$

The power series converges for all real and hence all complex z, so $f \in \mathscr{E}$, and clearly (1.1) holds for all real t. The uniqueness assertion in Proposition 1 is also clear, since entire functions that agree on the real axis are identical.

2.2. We turn now to the assertion about growth rates in Proposition 1. An inequality of Brelot and Choquet [10, Proposition 4] (or see [6, p. 80]) implies that

$$|H(\mathbf{e})| \leqslant \sqrt{d_m}\, r^{-m} M_2(H, r) \qquad (H \in \mathscr{H}_{m,N},\, r > 0), \tag{2.2}$$

where $d_m = \dim \mathscr{H}_{m,N}$. The spaces $\mathscr{H}_{m,N}$ are mutually orthogonal in the sense that

$$\int_S H_m(rx)\, H_n(rx)\, d\sigma(x) = 0 \qquad (H_m \in \mathscr{H}_{m,N},\, H_n \in \mathscr{H}_{n,N},\, m \neq n,\, r > 0)$$

(see, e.g., [6, p. 75]). Hence, if $\sum_{j=0}^{\infty} H_j$ is the polynomial expansion of a function $h \in \mathscr{H}_N$, then since the series converges uniformly on every sphere,

$$M_2^2(h, r) = \sum_{j=0}^{\infty} M_2^2(H_j, r) \qquad (r > 0). \tag{2.3}$$

From (2.2) and (2.3) we obtain

$$|H_m(\mathbf{e})| \leqslant \sqrt{d_m}\, r^{-m} M_2(h, r). \tag{2.4}$$

Suppose that $0 \leqslant \rho(h) < +\infty$. If $\varepsilon > 0$, then $M_2(h, r) = O(\exp(r^{\rho(h)+\varepsilon}))$. Since

$$\frac{d_m}{m^{N-2}} \to \frac{2}{(N-2)!} \qquad (m \to \infty) \tag{2.5}$$

(see, e.g., [6, p. 94]), it follows that there is a constant $C = C(h, \varepsilon, N)$ such that

$$|H_m(\mathbf{e})| \leqslant C m^{(N-2)/2} r^{-m} \exp(r^{\rho(h)+\varepsilon}) \qquad (m \geqslant 1, r > 0).$$

Taking $r = (m/(\rho(h)+\varepsilon))^{1/(\rho(h)+\varepsilon)}$, we obtain

$$|H_m(\mathbf{e})| \leqslant C m^{(N-2)/2} (e(\rho(h)+\varepsilon)/m)^{m/(\rho(h)+\varepsilon)},$$

which implies that

$$\limsup_{m \to \infty} \frac{m \log m}{\log(1/H_m(\mathbf{e}))} \leqslant \rho(h) + \varepsilon.$$

Hence, if f is given by (2.1), then $\rho(f) \leqslant \rho(h) + \varepsilon$ by the well-known formula for the order of an entire function in terms of its Taylor coefficients (see, e.g., [7, p. 9]). It follows that $\rho(f) \leqslant \rho(h)$.

It remains to show that if $0 < \rho(f) = \rho(h) < +\infty$, then $\tau(f) \leqslant \tau(h)$. Suppose that $0 \leqslant \tau(h) < +\infty$. If $\varepsilon > 0$, then $M_2(h, r) = O(\exp((\tau(h)+\varepsilon)\, r^{\rho(h)}))$. Hence, by (2.4) and (2.5), there is a constant $C = C(h, \varepsilon, N)$ such that

$$|H_m(\mathbf{e})| \leqslant C m^{(N-2)/2} r^{-m} \exp((\tau(h)+\varepsilon)\, r^{\rho(h)}) \qquad (m \geqslant 1, r > 0).$$

The choice $r = (m/((\tau(h)+\varepsilon)\, \rho(h)))^{1/\rho(h)}$ yields

$$|H_m(\mathbf{e})| \leqslant C m^{(N-2)/2} ((\tau(h)+\varepsilon)\, e\rho(h)/m)^{m/\rho(h)},$$

which implies that

$$\limsup_{m \to \infty} (m\, |H_m(\mathbf{e})|^{\rho(h)/m}) \leqslant (\tau(h)+\varepsilon)\, e\rho(h)$$

and hence that $\tau(f) \leqslant \tau(h) + \varepsilon$ (see, e.g., [7, p. 11]). Thus $\tau(f) \leqslant \tau(h)$.

2.3. The following elementary lemma is needed for the proof of Proposition 2.

LEMMA 1. *Let* (a_n) *be a sequence of non-negative numbers such that* $\sum_{n=1}^{\infty} a_n r^n < +\infty$ *for all* $r > 0$. *If*

$$\sum_{n=1}^{\infty} a_n r^n = O(r^p e^{\lambda r}) \qquad (r \to +\infty) \tag{2.6}$$

for some real number p and positive number λ, *then*

$$\sum_{n=1}^{\infty} n^q a_n r^n = O(r^{p+q} e^{\lambda r}) \qquad (r \to +\infty) \tag{2.7}$$

for each real number q. The same is true if O is replaced by o in both (2.6) *and* (2.7).

To prove the lemma, note first that the series $\sum_{n=1}^{\infty} a_n z^n$ converges to an entire function ϕ, say. Hence, if (2.6) holds, then by Cauchy's estimates

$$a_n \leqslant r^{-n} M_\infty(\phi, r) = r^{-n}\phi(r) \leqslant C r^{p-n} e^{\lambda r}$$

for all $n \geqslant 1$, $r > 1$ and some constant $C = C(\phi)$. If n is large enough, we can take $r = (n-p)/\lambda$ and obtain

$$a_n \leqslant C(\lambda e/(n-p))^{n-p} = O(n^p(\lambda e/n)^n). \tag{2.8}$$

We consider separately the cases $q > 0$ and $q < 0$. Suppose first that $q > 0$. Writing $R = \lambda e r$, we have by (2.8)

$$\begin{aligned}\sum_{n>4R} n^q a_n r^n &= O\Big(\sum_{n>4R} n^{p+q}(R/n)^n\Big)\\ &= O\Big(\sum_{n>4R} (2R/n)^n\Big)\\ &= O\Big(\sum_{n>4R} 2^{-n}\Big) = o(1)\end{aligned}$$

and

$$\sum_{1 \leqslant n \leqslant 4R} n^q a_n r^n \leqslant (4R)^q \sum_{n=1}^{\infty} a_n r^n = O(r^{p+q} e^{\lambda r}), \tag{2.9}$$

so that (2.7) holds. If (2.6) holds with o in place of O, then so also do (2.9) and (2.7).

Now suppose that $q<0$. Writing $p^+=\max\{0, p\}$, we have by (2.8)

$$\sum_{1\leqslant n<\lambda r/2} n^q a_n r^n = O\Bigg(\sum_{1\leqslant n<\lambda r/2} n^p(\lambda er/n)^n\Bigg)$$
$$= O(r^{1+p^+}(2e)^{\lambda r/2}), \qquad (2.10)$$

since the function $t\mapsto(\lambda er/t)^t$ increases on $(0, \lambda r]$. The expression in (2.10) is $o(e^{9\lambda r/10})$. Also

$$\sum_{n\geqslant\lambda r/2} n^q a_n r^n \leqslant (\lambda r/2)^q \sum_{n=1}^{\infty} a_n r^n = O(r^{p+q}e^{\lambda r}), \qquad (2.11)$$

so that (2.7) holds. If (2.6) holds with o in place of O, then so also do (2.11) and (2.7).

2.4. Here we prove the positive assertions in Proposition 2. Recall that if $\sum_{j=0}^{\infty} H_j$ is the polynomial expansion of a function $h\in\mathscr{H}_N$, then the entire function f satisfying (1.1) is given by (2.1), so that

$$M_2^2(f, r) = \sum_{j=0}^{\infty} (H_j(\mathbf{e}))^2\, r^{2j}.$$

Hence, by (2.4)

$$M_2^2(f, r) \leqslant \sum_{j=0}^{\infty} d_j M_2^2(H_j, r) = \sum_{j=0}^{\infty} d_j M_2^2(H_j, 1)\, r^{2j}. \qquad (2.12)$$

If (1.2) holds, then by (2.3)

$$\sum_{j=0}^{\infty} M_2^2(H_j, 1)\, r^{2j} = M_2^2(h, r) = O(r^{2p}e^{2\lambda r}). \qquad (2.13)$$

Since $d_j = O(j^{N-2})$, it follows from (2.12), (2.13), and Lemma 1 that

$$M_2(f, r) = O(r^{p+(N-2)/2}e^{\lambda r}). \qquad (2.14)$$

Now define $g=f^2$. By Cauchy's derivatives formula,

$$|g^{(n)}(0)| = \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{g(re^{i\theta})}{r^n e^{in\theta}}\, d\theta\right| \leqslant n!\, r^{-n} M_2^2(f, r).$$

Hence, by (2.14), there is a constant $C=C(h, N)$ such that

$$|g^{(n)}(0)| \leqslant Cn! r^{2p+N-2-n} e^{2\lambda r} \qquad (n\geqslant 0, r>1). \qquad (2.15)$$

If $n > 2\lambda$, we can take $r = n/(2\lambda)$ and obtain

$$g^{(n)}(0) = O(n^{2p+N-2} n!(2\lambda e/n)^n)$$
$$= O(n^{2p+N-3/2}(2\lambda)^n),$$

by Stirling's formula. Hence

$$M_\infty^2(f, r) = M_\infty(g, r)$$
$$\leqslant \sum_{n=0}^{\infty} |g^{(n)}(0)|\, r^n/n!$$
$$= O\left(\sum_{n=1}^{\infty} n^{2p+N-3/2}(2\lambda r)^n/n!\right)$$
$$= O(r^{2p+N-3/2}e^{2\lambda r}),$$

by Lemma 1, and (1.3) follows.

If (1.2) holds with o in place of O, then so also does (2.13) and hence so does (2.14), and the argument leading to (2.15) shows that if $\varepsilon > 0$, then

$$|g^{(n)}(0)| \leqslant \varepsilon n!\, r^{2p+N-2-n} e^{2\lambda r} \qquad (n \geqslant 0,\ r > r_o),$$

where r_o is independent of n. Hence if $n > 2\lambda r_o$, we can take $r = n/(2\lambda)$ and deduce that

$$g^{(n)}(0) = o(n^{2p+N-3/2}(2\lambda)^n),$$

which leads to the conclusion that (1.3) holds with o in place of O.

2.5. Here we give an example to show that if (1.2) holds, then (1.3) may fail with o in place of O. The Bessel function of the third kind of order v is denoted by I_v. If $x = (x_1, ..., x_N) \in \mathbb{R}^N$, then we write $x' = (x_2, ..., x_N)$. We define a function h on $\mathbb{R}^N$ by

$$h(x) = \begin{cases} \|x'\|^{(3-N)/2} I_{(N-3)/2}(\pi \|x'\|) \sin(\pi x_1) & (\|x'\| \neq 0) \\ c_N \sin(\pi x_1) & (\|x'\| = 0), \end{cases} \tag{2.16}$$

where $c_N = \lim_{t \to 0} t^{(3-N)/2} I_{(N-3)/2}(\pi t)$. (The limit exists and is positive; see, e.g., [23, p. 77].)

Example 1. Let h be given by (2.16). Then $h \in \mathscr{H}_N$ and

$$M_2(h, r) = O(r^{(3-2N)/4} e^{\pi r}). \tag{2.17}$$

The entire function f satisfying (1.1) is given by $f(z) = c_N \sin(\pi z)$ and $M_\infty(f, r) \neq o(e^{\pi r})$.

For the harmonicity of h, see [9, pp. 689–690]. Since $I_{-1/2}(t) = (2/\pi t)^{1/2} \cosh t$ [23, p. 80], we have in the case $N=2$

$$h(x_1, x_2) = (\sqrt{2}/\pi) \sin(\pi x_1) \cosh(\pi x_2).$$

Hence in this case

$$\begin{aligned} M_2^2(h, r) &= \pi^{-3} \int_0^{2\pi} \sin^2(\pi r \cos\theta) \cosh^2(\pi r \sin\theta)\, d\theta \\ &< 2^{-1}\pi^{-3} \int_0^{2\pi} (1 + \cosh(2\pi r \sin\theta))\, d\theta \\ &= \pi^{-2} + \pi^{-3} \int_0^{\pi} \cosh(2\pi r \sin\theta)\, d\theta \\ &= \pi^{-2}(1 + I_o(2\pi r)) \qquad [23, \text{p. } 79]. \end{aligned}$$

In the case where $N \geqslant 3$, we introduce polar coordinates in which $x_1 = \|x\| \cos\theta$ and find that

$$M_2^2(h, r) = \int_0^{\pi} \sin^{N-2}\theta (\psi(r \sin\theta) \sin(\pi r \cos\theta))^2\, d\theta \Big/ \int_0^{\pi} \sin^{N-2}\theta\, d\theta, \qquad (2.18)$$

where $\psi(t) = t^{(3-N)/2} I_{(N-3)/2}(\pi t)$. There is a constant $C = C(N)$ such that

$$0 < \psi(t) \leqslant C t^{(2-N)/2} e^{\pi t} \qquad (t > 0)$$

[23, p. 203]. Hence the integral in the numerator in (2.18) is

$$\begin{aligned} O\left(r^{2-N} \int_0^{\pi} e^{2\pi r \sin\theta}\, d\theta\right) &= O\left(r^{2-N} \int_0^{\pi} \cosh(2\pi r \sin\theta)\, d\theta\right) \\ &= O(r^{2-N} I_o(2\pi r)). \end{aligned}$$

Hence for all $N \geqslant 2$,

$$M_2(h, r) = O(r^{(2-N)/2} \sqrt{I_o(2\pi r)}) = O(r^{(3-2N)/4} e^{\pi r}) \qquad [23, \text{p. } 203].$$

The statements about the entire function f are obvious.

3. HARMONIC FUNCTIONS VANISHING AT LATTICE POINTS

We write $\mathbb{Z}$ for the set of all integers and $\mathbb{N}$ for the set of all non-negative integers.

THEOREM 1. *Suppose that* $h \in \mathscr{H}_N$ *and* $h(m\mathbf{e}) = 0$ *for all* $m \in \mathbb{Z}$.

(i) *If*

$$M_2(h, r) = O(r^p e^{\pi r}) \qquad (r \to +\infty) \tag{3.1}$$

for some $p \geqslant (3-2N)/4$, *then* $h(t\mathbf{e}) = P(t)\sin(\pi t)$ *for all real* t, *where* P *is a polynomial of degree at most* $p + (2N-3)/4$.

(ii) *If*

$$M_2(h, r) = o(r^{(3-2N)/4} e^{\pi r}) \qquad (r \to +\infty), \tag{3.2}$$

then $h(t\mathbf{e}) = 0$ *for all real* t.

Theorem 1(ii) may be compared with a result of N. V. Rao [20, Theorem 1.3], which states that if $h \in \mathscr{H}_N$ and h is of growth $(1, \tau)$ for some $\tau < \pi$ and if $h(m\mathbf{e}) = 0$ for all $m \in \mathbb{N}$, then $h(t\mathbf{e}) = 0$ and for all $t \in \mathbb{R}$. Clearly, Rao's growth hypothesis is more restrictive than (3.2). On the other hand, he supposes only that $h(m\mathbf{e}) = 0$ for all $m \in \mathbb{N}$, while we require $h(m\mathbf{e}) = 0$ for all $m \in \mathbb{Z}$. Indeed, in Rao's theorem it would suffice to have $h(m\mathbf{e}) = 0$ for all but finitely many $m \in \mathbb{N}$. In contrast, we shall give a simple example (Example 2 in Section 4.3) to show that Theorem 1 fails if we merely suppose that $h(m\mathbf{e}) = 0$ for all but finitely many $m \in \mathbb{Z}$.

In Example 1 we showed that the harmonic function h given by (2.16) satisfies (2.17), and obviously this function has the property that $h(m\mathbf{e}) = 0$ for all $m \in \mathbb{Z}$. Thus we have an example showing that Theorem 1(ii) is best possible in the sense that o cannot be replaced by O in (3.2). Our next theorem shows that there is a sense in which this example is unique. We shall say that an element h of $\mathscr{H}_N$ is x_1-*axial* if $h(x)$ depends only on x_1 and $\|x\|$ (equivalently, x_1 and $\|x'\|$).

THEOREM 2. *Suppose that* h *is an* x_1*-axial element of* $\mathscr{H}_N$ *such that* $h(m\mathbf{e}) = 0$ *for all* $m \in \mathbb{Z}$. *If*

$$M_2(h, r) = O(r^{(3-2N)/4} e^{\pi r}), \tag{3.3}$$

then h *is a constant multiple of the function given by* (2.16).

In particular, if $h \in \mathscr{H}_2$ *and* h *satisfies*

$$\begin{aligned} h(x_1, x_2) &= h(x_1, -x_2) && ((x_1, x_2) \in \mathbb{R}^2), \\ h(m, 0) &= 0 && (m \in \mathbb{Z}), \\ M_2(h, r) &= O(r^{-1/4} e^{\pi r}), \end{aligned} \tag{3.4}$$

then $h(x) = c\sin(\pi x_1)\cosh(\pi x_2)$ *for some constant* c.

Remark. We have stated Theorem 2 in a weak form in order to emphasise the contrast between (3.3) and (3.2). In fact (3.3) can be replaced by the milder condition

$$M_2(h, r) = o(r^{1+(3-2N)/4} e^{\pi r}), \tag{3.5}$$

and accordingly in (3.4) we can replace the right-hand side by $o(r^{3/4} e^{\pi r})$.

The remaining results in this section are mainly corollaries of Theorems 1 and 2.

THEOREM 3. *Suppose that* $h \in \mathscr{H}_N$ *and* (3.2) *holds. If* $h(x) = 0$ *when* $x \in \mathbb{Z}^{N-1} \times \{0, 1\}$, *then* $h \equiv 0$.

Theorem 3 was first proved in the case $N = 2$ by Boas [8, Theorem 1] under the stronger growth hypothesis that h is of growth $(1, \tau)$ with $\tau < \pi$. Zeilberger [25] and Rao [20] generalized Boas' theorem to $\mathbb{R}^N$. In fact Rao's result has a $(1, \tau)$ growth hypothesis, $\tau < \pi$, but supposes only that $h(x) = 0$ when $x \in \mathbb{N}^{N-1} \times \{0, 1\}$; see also [2].

In Theorem 3, again, the o condition (3.2) cannot be relaxed to O. To see this, note that the function given by (2.16) vanishes on $\mathbb{Z} \times \mathbb{R}^{N-1}$ and satisfies (2.17).

THEOREM 4. *Suppose that* $h \in \mathscr{H}_N$ *and* (3.2) *holds. If* $h(x) = (\partial h/\partial x_N)(x) = 0$ *whenever* $x \in \mathbb{Z}^{N-1} \times \{0\}$, *then* $h \equiv 0$.

Theorem 4 was proved by Zeilberger [25] under the growth hypothesis (neither stronger nor weaker than ours) that $|h(x)| < C \exp(\tau(|x_1| + \cdots + |x_N|))$ for all $x \in \mathbb{R}^N$ and some constants C and $\tau < \pi$. A result of the same type with $\mathbb{N}^{N-1} \times \{0\}$ in place of $\mathbb{Z}^{N-1} \times \{0\}$ but with a more restrictive growth hypothesis is given in [2, Theorem 4].

Again the function in (2.16) shows that the growth condition (3.2) in Theorem 4 is sharp, for this function vanishes, together with its x_N-derivative, on $\mathbb{Z} \times \mathbb{R}^{N-1}$.

THEOREM 5. *Suppose that* $h \in \mathscr{H}_N$ *and* (3.2) *holds. If* $h(x) = (-1)^{N+1} \times h(-x)$ *for each* $x \in \mathbb{R}^N$ *and* $h(x) = 0$ *whenever* $x \in \mathbb{Z}^N$ *and* $x_j = 0$ *for some* $j \in \{1, ..., N\}$, *then* $h \equiv 0$.

Theorem 5 was proved by Ching [13] in the case $N = 2$ under the more restrictive growth hypothesis that h is of growth $(1, \tau)$ for some $\tau < \pi$. Ching's result was generalized to $\mathbb{R}^N (N \geqslant 2)$ and improved ($\mathbb{N}^N$ replacing $\mathbb{Z}^N$) in [2].

We shall give an example (Example 3 in Section 4.10) to show that the growth condition (3.2) in Theorem 5 cannot be relaxed to a O condition.

THEOREM 6. *Suppose that* $h \in \mathscr{H}_2$ *and*

$$M_2(h, r) = o(r^{3/4} e^{\pi r}). \tag{3.6}$$

If $h(m, 0) = h(m, 1) = 0$ *for each integer* m, *then* h *is given by*

$$h(x_1, x_2) = \sin(\pi x_1)(\alpha_1 e^{\pi x_2} + \alpha_2 e^{-\pi x_2}) + \sin(\pi x_2)(\alpha_3 e^{\pi x_1} + \alpha_4 e^{-\pi x_1}),$$

where $\alpha_1, ..., \alpha_4$ *are constants.*

The question as to whether (3.6) can be relaxed remains open.

4. PROOFS OF THEOREMS 1–6; EXAMPLES

4.1. Theorem 1 follows easily from Proposition 2 and the following classical result.

LEMMA 2. *Suppose that* $f \in \mathscr{E}$ *and* $f(m) = 0$ *for all* $m \in \mathbb{Z}$. *If*

$$M_\infty(f, r) = O(r^q e^{\pi r}) \tag{4.1}$$

for some $q \geqslant 0$, *then* $f(z) = P(z) \sin(\pi z)$, *where* P *is a polynomial of degree at most* q. *If* (4.1) *holds with* o *in place of* O, *then the degree of* P *is less than* q *in the case* $q > 0$, *and* $f \equiv 0$ *in the case* $q = 0$.

For the first statement in Lemma 2, see [7, p. 156]. The o results are simple consequences of the O result.

4.2. If h satisfies the hypotheses of Theorem 1(i), then by Proposition 2, the entire function f associated to h satisfies

$$M_\infty(f, r) = O(r^{p + (2N-3)/4} e^{\pi r}).$$

Hence by Lemma 2, $f(z) = P(z) \sin(\pi z)$, where P is a polynomial of degree at most $p + (2N-3)/4$, and the conclusion of Theorem 1(i) follows.

Theorem 1(ii) is proved in the same way, using Proposition 2 and the o-form of Lemma 2 with $q = 0$.

4.3. The following example shows that in Theorem 1 it is not enough to suppose that $h(m\mathbf{e}) = 0$ for all but finitely many integers m.

EXAMPLE 2. Let k be a positive integer and define h on $\mathbb{R}^N$ by

$$h(x) = \operatorname{Re}\left(\sin(\pi\zeta) \prod_{j=1}^{k} (\zeta - j)^{-1} \right),$$

where $\zeta = x_1 + ix_2$. (At points x for which $\zeta \in \{1, ..., k\}$, we define h by continuous extension.) Then $h \in \mathscr{H}_N$, and $M_2(h, r) = O(r^{-k}e^{\pi r})$. Also, $h(m\mathbf{e}) = 0$ for all $m \in \mathbb{Z} \backslash \{1, ..., k\}$. However, $h(t\mathbf{e})$ is not of the form $P(t)\sin(\pi t)$ with P a polynomial.

We omit the straightforward verification.

4.4. Here we prove Theorem 2 with the milder hypothesis (3.5) in place of (3.4). By Proposition 2, if $h \in \mathscr{H}_N$ and (3.5) holds, then the entire function f associated to h satisfies $M_\infty(f, r) = o(re^{\pi r})$. Hence by the o-form of Lemma 2, if $h(m\mathbf{e}) = 0$ for each $m \in \mathbb{Z}$, then $h(t\mathbf{e}) = f(t) = c\sin(\pi t)$ for some constant c and all real t. The function given by (2.16) therefore agrees with h, up to a multiplicative constant, on the x_1-axis. Hence, to complete the proof of Theorem 2, it suffices to prove the following lemma.

Lemma 3. *If g is an x_1-axial element of $\mathscr{H}_N$ and $g(t\mathbf{e}) = 0$ for all real t, then $g \equiv 0$.*

We write $D^j = \partial^j/\partial x_N^j$ for $j \in \mathbb{N}$. If $D^j g$ vanishes identically on the x_1-axis for all j, then since the Taylor series of g about any point converges to g on $\mathbb{R}^N$, it follows that $g(x_1, 0, ..., 0, x_N) = 0$ for all x_1, x_N and hence $g \equiv 0$, by axial symmetry.

Now suppose that there is a least k such that $D^k g$ is not identically zero on the x_1-axis. Note that k must be even, for the axial symmetry implies that $D^j g = 0$ on the x_1-axis when j is odd. We may suppose that there exist a bounded open interval J and a number $\delta > 0$ such that $D^k g(t\mathbf{e}) > k!\,\delta$ for all $t \in J$; otherwise consider $-g$. By Taylor's theorem, if $t \in J$ and $-1 < s < 1$, then

$$g(t, 0, ..., 0, s) \geqslant \delta s^k - C\,|s|^{k+1},$$

where C does not depend on t or s. Hence $g(t, 0, ..., 0, s) \geqslant 0$ when $t \in J$ and $|s|$ is sufficiently small. Hence, by symmetry, $g \geqslant 0 = g(y)$ on some ball centred at a point y on the x_1-axis. It follows from the minimum principle that $g \equiv 0$.

4.5. To prove Theorems 3,4,5, we need to know that the growth condition (3.2) is unaffected by a translation of axes. Let $M_2(h, x, r)$ denote the square-root of the mean value of h^2 on the sphere of centre x and radius r (so that $M_2(h, 0, r) = M_2(h, r)$).

Lemma 4. *If $h \in \mathscr{H}_N$ and*

$$M_2(h, x, r) = o(r^p e^{\lambda r}) \qquad (r \to +\infty),$$

where p, λ are real numbers, $\lambda \geqslant 0$, holds with $x = 0$, then it holds for all $x \in \mathbb{R}^N$.

We use a result about integrals of subharmonic functions. A special case of [1, Theorem 1] implies that if u is a non-negative subharmonic function on $\mathbb{R}^N$ and S_1, S_2 are spheres with S_1 contained in the closed ball bounded by S_2, then the surface integral of u over S_1 is no greater than twice the surface integral of u over S_2. Taking $u = h^2$ and expressing this result in terms of means values, we obtain

$$r^{N-1} M_2^2(h, x, r) \leqslant 2(\|x\| + r)^{N-1} M_2^2(h, 0, \|x\| + r),$$

from which the lemma follows.

4.6. The next lemma is also used in the proofs of Theorems 3, 4, 5.

LEMMA 5. *Suppose that $h \in \mathscr{H}_N$ and (3.2) holds. If $a \in \mathbb{R}$ and $h = 0$ on $\mathbb{Z}^{N-1} \times \{a\}$, then $h = 0$ on $\mathbb{R}^{N-1} \times \{a\}$.*

By Lemma 4, the growth hypothesis (3.2) is unaffected by a translation of axes, so we may suppose that $a = 0$. By Theorem 1, $h = 0$ on the x_1-axis. Similarly, by a translation, $h = 0$ on every line of the form $\{(t, x_2, ..., x_{N-1}, 0): t \in \mathbb{R}\}$, where $x_2, ..., x_{N-1} \in \mathbb{Z}$. Thus $h = 0$ on $\mathbb{R} \times \mathbb{Z}^{N-2} \times \{0\}$. Fix $x_1 \in \mathbb{R}$ and $x_3, ..., x_{N-1} \in \mathbb{Z}$. Then $h(x_1, m, x_3, ..., x_{N-1}, 0) = 0$ for each $m \in \mathbb{Z}$. Hence by Theorem 1 (with a translation and rotation of axes), $h(x_1, t, x_3, ..., x_{N-1}, 0) = 0$ for all real t. Thus $h = 0$ on $\mathbb{R}^2 \times \mathbb{Z}^{N-3} \times \{0\}$. Proceeding inductively, we obtain that $h = 0$ on $\mathbb{R}^{N-1} \times \{0\}$, as required.

4.7. We can now complete the proof of Theorem 3. By Lemma 5, $h = 0$ on $\mathbb{R}^{N-1} \times \{0, 1\}$. By repeated use of the reflection principle, $h = 0$ on $\mathbb{R}^{N-1} \times \mathbb{Z}$. Fix $x_1, ..., x_{N-1} \in \mathbb{R}$. Then $h(x_1, ..., x_{N-1}, m) = 0$ for each $m \in \mathbb{Z}$, and Theorem 1, together with Lemma 4, implies that $h = 0$ on the line $\{(x_1, ..., x_{N-1}, t): t \in \mathbb{R}\}$. Since $x_1, ..., x_{N-1}$ are arbitrary, we have $h \equiv 0$.

4.8. The proof of Theorem 4 requires the following lemma.

LEMMA 6. *If $h \in \mathscr{H}_N$ and h satisfies the growth condition*

$$M_2(h, r) = o(r^p e^{\lambda r}) \qquad (r \to +\infty),$$

where p, λ are real numbers, $\lambda > 0$, then so also does $\partial h / \partial x_N$.

Let $\sum_{j=0}^{\infty} H_j$ be the polynomial expansion of h. Then the polynomial expansion of $\partial h / \partial x_N$ is $\sum_{j=1}^{\infty} (\partial H_j / \partial x_N)$. An inequality of Calderón and Zygmund [12, formula (1.5.2), Chap. I, Sects. 7, 8] (or see Kuran [18, Theorem 4 and formula (5)]) shows that

$$M_2(\partial H_j / \partial x_N, r) \leqslant r^{-1} j \sqrt{d_j / d_{j-1}}\, M_2(H_j, r) \qquad (j \geqslant 1, r > 0).$$

Since $d_j/d_{j-1}=O(1)$ (see, e.g., the formula for d_j in [6, p. 82]), we obtain (cf. (2.3))

$$\begin{aligned} M_2^2(\partial h/\partial x_N, r) &= \sum_{j=1}^{\infty} M_2^2(\partial H_j/\partial x_N, r) \\ &= O\left(r^{-2}\sum_{j=1}^{\infty} j^2 M_2^2(H_j, r)\right) \\ &= O\left(r^{-2}\sum_{j=1}^{\infty} j^2 M_2^2(H_j, 1)\, r^{2j}\right). \end{aligned} \tag{4.2}$$

Now

$$\sum_{j=0}^{\infty} M_2^2(H_j, 1)\, r^{2j} = M_2^2(h, r) = o(r^{2p} e^{2\lambda r}),$$

so the conclusion follows from (4.2) and the o-form of Lemma 1.

4.9. Theorem 4 now follows easily. Lemmas 5 and 6 imply that $h=\partial h/\partial x_N=0$ on $\mathbb{R}^{N-1}\times\{0\}$. Writing $\bar{x}=(x_1, ..., x_{N-1}, -x_N)$, we have, by the reflection principle, $h(\bar{x})=-h(x)$ and $(\partial h/\partial x_N)(\bar{x})=-(\partial h/\partial x_N)(x)$ for all $x\in\mathbb{R}^N$, and these equations imply that $h\equiv 0$.

4.10. If the hypotheses of Theorem 5 are satisfied, then by Lemma 5, $h=0$ on $\{x\in\mathbb{R}^N: x_N=0\}$. Similarly, $h=0$ on each of the hyperplanes $\{x\in\mathbb{R}^N: x_j=0\}$ for $j=1, ..., N$. The reflection principle applied to each of these hyperplanes yields $h(x)=(-1)^N h(-x)$ for each $x\in\mathbb{R}^N$. Since, by hypothesis, $h(x)=(-1)^{N+1} h(-x)$, we have $h\equiv 0$.

4.11. Here is an example to show that in Theorem 5 the o in (3.2) cannot be relaxed to O.

EXAMPLE 3. Let h be the function given by (2.16) and define g on $\mathbb{R}^N$ by

$$g=h \quad (N \text{ even}), \qquad g=\partial h/\partial x_N \quad (N \text{ odd}).$$

Then $g\in\mathscr{H}_N$, $g(x)=(-1)^{N+1} g(-x)$ for each $x\in\mathbb{R}^N$, $g=0$ on $\mathbb{Z}^N$ and

$$M_2(g, r)=O(r^{(3-2N)/4} e^{\pi r}). \tag{4.3}$$

Clearly the function h in (2.16) satisfies $h(x)=-h(-x)$ and hence $(\partial h/\partial x_N)(x)=(\partial h/\partial x_N)(-x)$ for each $x\in\mathbb{R}^N$. Thus $g(x)=(-1)^{N+1} g(-x)$. Since h vanishes on $\mathbb{Z}\times\mathbb{R}^{N-1}$, so also does $\partial h/\partial x_N$. Thus $g=0$ on $\mathbb{Z}\times\mathbb{R}^{N-1}$. Finally, we have seen that h satisfies the growth condition (2.17). A simple modification (replacing o by O) of Lemma 6 shows that $\partial h/\partial x_N$ satisfies the same growth condition. Thus (4.3) holds.

4.12. Suppose that h satisfies the hypotheses of Theorem 6. By Proposition 2, the associated entire function f satisfies $M_\infty(f, r) = o(re^{\pi r})$. Hence, by the o-form of Lemma 2 with $q = 1$, we have $f(z) = a_1 \sin(\pi z)$ for some constant a_1. Thus $h(t, 0) = a_1 \sin(\pi t)$ for all real t. Similarly, using Lemma 4, we obtain $h(t, 1) = a_2 \sin(\pi t)$ for some constant a_2 and all real t. Define g on $\mathbb{R}^2$ by

$$g(x_1, x_2) = \sin(\pi x_1)(\alpha_1 e^{\pi x_2} + \alpha_2 e^{-\pi x_2}),$$

where α_1, α_2 are chosen so that $g = h$ on $\mathbb{R} \times \{0, 1\}$. Then $g - h \in \mathscr{H}_2$, and by repeated reflection, $g - h = 0$ on $\mathbb{R} \times \mathbb{Z}$. Also,

$$M_2(g, r) \leqslant M_\infty(g, r) = O(e^{\pi r}),$$

so $M_2(g - h, r) = o(r^{3/4} e^{\pi r})$. Fix $t \in \mathbb{R}$. Then $(g - h)(t, k) = 0$ for all $k \in \mathbb{Z}$. Hence by Lemma 4, Proposition 2, and Lemma 2, $(g - h)(t, s) = \psi(t) \sin(\pi s)$ for all $s \in \mathbb{R}$, where $\psi(t)$ is a number depending on t. The harmonicity of $g - h$ implies that ψ is a real-analytic function satisfying $\psi'' = \pi^2 \psi$. Hence $\psi(t)$ is a linear combination of $e^{\pi t}$ and $e^{-\pi t}$, so

$$(h - g)(x_1, x_2) = \sin(\pi x_2)(\alpha_3 e^{\pi x_1} + \alpha_4 e^{-\pi x_1})$$

for some constants α_3, α_4, as required.

5. RADIAL DECAY OF HARMONIC FUNCTIONS

It has been known for a very long time that there exist non-constant entire functions F satisfying $F(re^{i\theta}) \to 0$ as $r \to +\infty$ for all real θ. (Lindelöf's book [19, p. 122] of 1905 contains an example of such a function F. For an elementary construction see Burckel's recent article [11].) The real part of F is an element of $\mathscr{H}_2 \backslash \{0\}$ which also tends to 0 along every ray emanating from the origin, and there also exist elements of $\mathscr{H}_N \backslash \{0\}$ $(N \geqslant 2)$ exhibiting the same limiting behaviour. Using modern results on harmonic approximation [4] (or see [15, Chap. 5]), it is possible to produce elements of $\mathscr{H}_N \backslash \{0\}$ that decay very rapidly on rays: indeed, if $\mu > 0$, then there exists $h \in \mathscr{H}_N \backslash \{0\}$ such that

$$h(rx) = o(\exp(-r^\mu)) \qquad (r \to +\infty) \tag{5.1}$$

for all $x \in S$ [5, Example 1]. Moreover, arbitrarily rapid decay can occur on almost all rays: if ε: $[0, +\infty) \to (0, +\infty)$ is a decreasing function, then there exists $h \in \mathscr{H}_N \backslash \{0\}$ such that $h(rx) = o(\varepsilon(r))$ for σ-almost all $x \in S$ [5, Example 3]. For related results about functions in $\mathscr{E}$ or $\mathscr{H}_2$, see [21].

The following theorem shows that harmonic functions of finite order cannot exhibit such extreme behaviour.

THEOREM 7. *Suppose that* $h \in \mathscr{H}_N$ *and* $\rho(h) < \mu < +\infty$. *If* (5.1) *holds for some* $x \in S$, *then* $h(tx) = 0$ *for all real* t.

For harmonic functions of less than exponential growth the radial decay hypothesis (5.1) in Theorem 7 can be greatly relaxed.

THEOREM 8. *Suppose that* $h \in \mathscr{H}_N$ *and* h *is of growth* $(1, 0)$. *If*

$$h(rx) = O(r^p) \qquad (r \to +\infty) \tag{5.2}$$

for some $x \in S$ *and some* $p > 0$, *then the function* $t \mapsto h(tx)$ *is a polynomial of degree at most* p. *If* $h(rx) \to 0$ *for some* $x \in S$, *then* $h(tx) = 0$ *for all real* t.

We shall call a subset E of S a *set of harmonic determination* (or SHD) if the only element h of $\mathscr{H}_N$ satisfying $h(tx) = 0$ for all $t \in \mathbb{R}$ and all $x \in E$ is identically zero. The following proposition gives some sufficient conditions for a set to be a SHD.

PROPOSITION 3. *A subset* E *of* S *is a SHD if any of the following conditions is satisfied*:

(i) $\bar{E}$ *has non-empty interior relative to* S,

(ii) $\sigma(\bar{E}) > 0$,

(iii) $\bar{E}$ *contains a set of the form* $\{x \in S : x_1 = \alpha\}$ *for some transcendental number* $\alpha \in (-1, 1)$.

In (iii) the condition that α be transcendental can be relaxed; the precise condition that α must satisfy will become apparent in the proof of Proposition 3 (Section 6.5).

Theorems 7 and 8 have respectively the following corollaries.

COROLLARY 1. *Suppose that* $h \in \mathscr{H}_N$ *and* $\rho(h) < +\infty$. *Let* E *be a SHD. If, for each* $x \in E$, *there is a number* $\mu(x) > \rho(h)$ *such that*

$$h(rx) = o(\exp(-r^{\mu(x)})),$$

then $h \equiv 0$.

COROLLARY 2. *Suppose that* $h \in \mathscr{H}_N$ *and* h *is of growth* $(1, 0)$. *Let* E *be a SHD.*

(i) *If* (5.2) *holds for all* $x \in E$ *and some* $p > 0$, *then* h *is a polynomial of degree at most* p.

(ii) *If* $h(rx)=o(r)$ *for all* $x\in E$ *and* $h(rx)=o(1)$ *for some* $x\in S$, *then* $h\equiv 0$.

A very simple example (Subsection 6.4) will show that Theorem 8 and Corollary 2 fail for harmonic functions of growth $(1,\varepsilon)$, where $\varepsilon>0$.

6. PROOFS OF THEOREMS 7, 8 AND PROPOSITION 3

6.1. Theorems 7 and 8 are simple consequences of Proposition 1 and classical results for holomorphic functions. In proving these theorems we may assume that $x=\mathbf{e}$.

If the hypotheses of Theorem 7 hold, then by Proposition 1, the entire function f associated to h is of order $\rho(f)\leqslant\rho(h)$, and f also satisfies

$$\min_{|z|=r}|f(z)|\leqslant|f(r)|=|h(r\mathbf{e})|=o(\exp(-r^{\mu})),$$

but this is impossible unless $f\equiv 0$ [7, p. 22]. Hence $h(t\mathbf{e})=0$ for all real t.

6.2. If the hypotheses of Theorem 8 are satisfied (with $x=\mathbf{e}$), then the associated entire function f is of growth $(1,0)$. By Hadamard's factorization theorem, either f is a polynomial or f has infinitely many zeros. It is enough to prove that if f has more than p zeros, then $f\equiv 0$. Suppose then that $z_1,\dots,z_n$ are zeros of f, where $n>p$. Define $F(z)=f(z)\times\Pi_{j=1}^{n}(z-z_j)^{-1}$. Then F is an entire function of growth $(1,0)$ and

$$|F(r)|=O(r^{-n}\,|h(r\mathbf{e})|)=o(1).$$

Hence $f=F\equiv 0$ ([7, p. 84]).

6.3. Corollary 1 follows immediately from Theorem 7.

Suppose that the hypotheses of Corollary 2(i) are satisfied. Let $\sum_{j=0}^{\infty}H_j$ be the polynomial expansion of h. If $x\in E$, then by Theorem 8 there is a polynomial P_x of degree at most p such that

$$P_x(t)=h(tx)=\sum_{j=0}^{\infty}H_j(x)\,t^j\qquad(t\in\mathbb{R}).$$

It follows from the uniqueness of power series expansions that $H_j(x)=0$ when $j>p$. Hence for such j we have $H_j(tx)=0$ for all $t\in\mathbb{R}$ and all $x\in E$. Since E is a SHD, it follows that $H_j\equiv 0$ when $j>p$, so that $h=\sum_{j\leqslant p}H_j$.

Corollary 2(ii) is easily deduced from Corollary 2(i).

6.4. The following example shows that in Theorem 8 and Corollary 2, growth $(1,0)$ cannot be replaced by growth $(1,\varepsilon)$ with $\varepsilon>0$. We omit the simple verification.

Example 4. Define h on $\mathbb{R}^N$ by

$$h(x)=e^{-\varepsilon x_1}\cos(\varepsilon x_2),$$

where $\varepsilon>0$. Then $h\in\mathcal{H}_N$ and h is of growth $(1,\varepsilon)$. If $x\in S$ and $x_1>0$, then $h(rx)\to 0$ as $r\to+\infty$, but $h\not\equiv 0$.

6.5. It remains to prove Proposition 3. Suppose that $h\in\mathcal{H}_N$ and $h(tx)=0$ for all $t\in\mathbb{R}$ and all $x\in E$. By continuity, $h=0$ on the set $\{tx\colon x\in\bar{E},\ t\in\mathbb{R}\}=E^*$, say. We have to show that if E satisfies (i), (ii), or (iii), then $h\equiv 0$.

If (i) holds, then E^* contains a non-empty open subset of $\mathbb{R}^N$ on which $h=0$, and hence $h\equiv 0$.

If (ii) holds, then E^* has positive (in fact, infinite) N-dimensional measure. It follows that all first-order partial derivatives of h vanish at all points of density of E^* [16, Lemma 7] and hence almost everywhere on E^* [24, Theorem 7.13]. Proceeding inductively, we find that all the partial derivatives of h vanish almost everywhere on E^*. Since h is real-analytic, we have $h=0$ on some neighbourhood of any point where all the partial derivatives vanish. Hence $h\equiv 0$.

Suppose that (iii) holds. We may suppose also that $0<\alpha<1$. Let C_α denote the cone $\{x\in\mathbb{R}^N\colon x_1=\alpha\,\|x\|\}$. In [3] it was shown in the case $N\geqslant 3$ that there exists $h\in\mathcal{H}_N\backslash\{0\}$ such that $h=0$ on C_α if and only if there exists a positive integer n such that α is a zero of the ultraspherical polynomial $P_n^{((N-2)/2)}$ (notation of [22]) or a zero of one of the derivatives of this polynomial. Hence there is certainly no such h if α is transcendental. In the case $N=2$ the set C_α consists of two lines which meet at an angle $2\cos^{-1}\alpha$, and there exists $h\in\mathcal{H}_2\backslash\{0\}$ such that $h=0$ on C_α if and only if $\cos^{-1}\alpha$ is a rational multiple of π. Hence, again, if α is transcendental, then there is no such h.

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